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A note on the stability of steady inviscid helical gas flows

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A necessary condition for linear stability of steady inviscid helical gas flows is found by the generalized progressing-wave expansion method. The criterion obtained is compared with the known Richardson number criteria giving sufficient conditions for stability.

1. Introduction

The stability of steady inviscid helical flows in an incompressible fluid has previously been studied by Howard & Gupta (1962) and others. More recently the same problem has been studied for inviscid compressible fluids by Howard (1973), Gans (1975) and Warren (1975). These authors obtained sufficient conditions for stability under various restrictions.

In this paper we study the problem considered by Warren (1975). By a different method we obtain a criterion giving necessary conditions for linear stability of the flows. Our method is based on the generalized progressing-wave expansion method described by Friedlander (1958) and Ludwig (1960). In fact, it is shown by Eckhoff (1975) that necessary conditions for stability may be obtained by investigating the stability properties of the leading terms in such expansions. A brief description of the generalized wave expansion method and its application to stability problems is given in an appendix.

As should be expected, it is possible to show that the known criteria giving sufficient conditions for stability are more restrictive in general than the criterion we obtain, which gives necessary conditions for stability. However, the criteria are shown to coincide in some cases; thus conditions which are both necessary and sufficient for stability are established for these cases.

2. The basic equations

The fundamental equations are

$$\partial \mathbf{v}/\partial t + \mathbf{v}\,.\,\nabla \mathbf{v} = -\rho^{-1}\nabla p + \nabla V, \tag{2.1a}$$

$$\partial \rho/\partial t + \mathbf{v}\,.\,\nabla \rho + \rho \nabla\,.\,\mathbf{v} = 0, \tag{2.1b}$$

$$\partial (p\rho^{-\gamma})/\partial t + \mathbf{v}\,.\,\nabla (p\rho^{-\gamma}) = 0, \tag{2.1c}$$

where $\mathbf{v}$ denotes the velocity, ρ the density, p the pressure, V a given potential for the external forces acting on the fluid and γ is a constant.

We let (r, ϕ, z) denote cylindrical co-ordinates where the z axis coincides with the axis of the two coaxial circular cylinders (radii a and b, $0 < a < b$) between which the flow occurs. The assumption $a > 0$ is introduced here only to avoid the singularities on the axis caused by the cylindrical co-ordinates. The results we obtain are in fact also valid when the inner cylinder is absent. The potential for the external forces is assumed to depend on r only, i.e. $V = V(r)$. The basic flow may then be written as

$$\mathbf{v} = v_0(r)\,\hat{\boldsymbol{\phi}} + w_0(r)\,\hat{\mathbf{z}}, \quad \rho = \rho_0(r), \quad p = p_0(r). \tag{2.2}$$

Here v_0, w_0 and ρ_0 may be chosen arbitrarily and p_0 is then determined to within an arbitrary additive constant by

$$p_0' = \rho_0(r^{-1}v_0^2 + V'), \tag{2.3}$$

where a prime denotes differentiation with respect to r.

In order to study the stability properties of the basic flow (2.2), we perturb it by introducing into (2.1) the following expressions:

$$\left.\begin{aligned} \mathbf{v} &= \rho_0^{-\frac{1}{2}} u_r \hat{\mathbf{r}} + (\rho_0^{-\frac{1}{2}} u_\phi + v_0)\,\hat{\boldsymbol{\phi}} + (\rho_0^{-\frac{1}{2}} u_z + w_0)\,\hat{\mathbf{z}}, \\ \rho &= \rho_0 + c_0^{-1}\rho_0^{\frac{1}{2}}(s_1 + s_2), \quad p = p_0 + c_0\rho_0^{\frac{1}{2}} s_2. \end{aligned}\right\} \tag{2.4}$$

Here $c_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ denotes the local sound speed and $\boldsymbol{\omega} = \{u_r, u_\phi, u_z, s_1, s_2\}$ represents the perturbation superimposed on the basic flow (2.2). The transformation (2.4) is analogous to transformations considered earlier by various authors (see Eckart 1960, p. 55; Yih 1965, p. 5). By substituting (2.4) into (2.1), the linearized equations for the perturbations are found to be

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{A}^1 \frac{\partial \boldsymbol{\omega}}{\partial r} + \mathbf{A}^2 \frac{\partial \boldsymbol{\omega}}{\partial \phi} + \mathbf{A}^3 \frac{\partial \boldsymbol{\omega}}{\partial z} + \mathbf{B}\boldsymbol{\omega} = 0. \tag{2.5}$$

Here $\boldsymbol{\omega}$ is treated as a column vector and the coefficient matrices are

$$\mathbf{A}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c_0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} r^{-1}v_0 & 0 & 0 & 0 & 0 \\ 0 & r^{-1}v_0 & 0 & 0 & r^{-1}c_0 \\ 0 & 0 & r^{-1}v_0 & 0 & 0 \\ 0 & 0 & 0 & r^{-1}v_0 & 0 \\ 0 & r^{-1}c_0 & 0 & 0 & r^{-1}v_0 \end{pmatrix}, \tag{2.6a, b}$$

$$\mathbf{A}^3 = \begin{pmatrix} w_0 & 0 & 0 & 0 & 0 \\ 0 & w_0 & 0 & 0 & 0 \\ 0 & 0 & w_0 & 0 & c_0 \\ 0 & 0 & 0 & w_0 & 0 \\ 0 & 0 & c_0 & 0 & w_0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -2r^{-1}v_0 & 0 & \beta & G \\ r^{-1}v_0 + v_0' & 0 & 0 & 0 & 0 \\ w_0' & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6c, d}$$

where

$$\left.\begin{aligned} \alpha &= c_0\rho_0^{-1}\rho_0' - c_0^{-1}(r^{-1}v_0^2 + V'), \quad \beta = -c_0^{-1}(r^{-1}v_0^2 + V'), \\ G &= c_0^{-1}(\tfrac{1}{2}\gamma - 1)(r^{-1}v_0^2 + V'), \quad H = r^{-1}c_0 + c_0^{-1}(r^{-1}v_0^2 + V') - \tfrac{1}{2}c_0\rho_0^{-1}\rho_0'. \end{aligned}\right\} \tag{2.7}$$

We see that (2.5) is a symmetric hyperbolic system. The characteristic equation associated with (2.5) is (see appendix)

$$\det\{\xi^1\mathbf{A}^1+\xi^2\mathbf{A}^2+\xi^3\mathbf{A}^3-\lambda\mathbf{I}\}$$
$$= (r^{-1}\xi^2v_0+\xi^3w_0-\lambda)^3[(r^{-1}\xi^2v_0+\xi^3w_0-\lambda)^2-c_0^2k^2] = 0, \quad (2.8)$$

where
$$k = \{(\xi^1)^2+(r^{-1}\xi^2)^2+(\xi^3)^2\}^{\frac{1}{2}}. \quad (2.9)$$

Thus the characteristic roots are seen to be

$$\Omega_1 = r^{-1}\xi^2v_0+\xi^3w_0, \quad (2.10a)$$

$$\Omega_2 = r^{-1}\xi^2v_0+\xi^3w_0+kc_0, \quad (2.10b)$$

$$\Omega_3 = r^{-1}\xi^2v_0+\xi^3w_0-kc_0. \quad (2.10c)$$

The two simple roots Ω_2 and Ω_3 correspond to the acoustic waves, while the triple root Ω_1 corresponds to the internal gravity waves (inertial waves). It is not expected that the acoustic waves will give rise to any instabilities of the basic flow (2.2), therefore we shall limit our discussion to the gravity waves.

It is possible to study the acoustic waves by the same method as we are going to apply to the gravity waves, but some additional complications then arise. In fact, for the acoustic waves phenomena such as focusing, diffraction and reflexion appear in the leading term of the expansion (see Friedlander 1958; Eckhoff 1975). If these problems are treated properly, it is possible to show that the leading term of the expansion for the acoustic waves is always stable. However, since we are going to establish only *necessary* conditions for stability, this lengthy discussion is of no concern.

The ray equations for the gravity waves associated with the characteristic root Ω_1 are (see appendix)

$$dr/dt = 0, \quad d\phi/dt = r^{-1}v_0, \quad dz/dt = w_0, \quad (2.11a–c)$$

$$d\xi^1/dt = r^{-1}\xi^2(r^{-1}v_0-v_0')-\xi^3w_0', \quad (2.11d)$$

$$d\xi^2/dt = 0, \quad d\xi^3/dt = 0. \quad (2.11e,f)$$

The solutions of (2.11) are readily found to be

$$r = r_0, \quad \phi = \phi_0+r_0^{-1}v_0(r_0)\,t, \quad z = z_0+w_0(r_0)\,t, \quad (2.12a–c)$$

$$\xi^1 = \xi_0^1+\{r_0^{-1}\xi_0^2[r_0^{-1}v_0(r_0)-v_0'(r_0)]-\xi_0^3w_0'(r_0)\}\,t, \quad (2.12d)$$

$$\xi^2 = \xi_0^2, \quad \xi^3 = \xi_0^3, \quad (2.12e,f)$$

where r_0, ϕ_0, z_0, ξ_0^1, ξ_0^2 and ξ_0^3 denote the initial values at $t = 0$. Thus we see that the rays for the gravity waves coincide with the streamlines of the basic flow. This implies that the leading term of the generalized progressing-wave expansion for these waves is not affected by the presence of boundaries (see appendix and Eckhoff 1975, p. 78).

The amplitude of the leading term of the generalized progressing-wave expansion for the gravity waves is given by (see appendix)

$$\mathbf{a}_0 = \sigma_1\mathbf{r}_1+\sigma_2\mathbf{r}_2+\sigma_3\mathbf{r}_3 \quad (2.13)$$

along the rays (2.12). Here $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are linearly independent eigenvectors associated with the triple characteristic root Ω_1, and σ_1, σ_2 and σ_3 are determined by the so-called transport equations. It is not difficult to see that here we may choose

$$\mathbf{r}_1 = k^{-1}\{r^{-1}\xi^2, -\xi^1, 0, \xi^3, 0\}, \tag{2.14a}$$

$$\mathbf{r}_2 = k^{-1}\{0, \xi^3, -r^{-1}\xi^2, \xi^1, 0\}, \tag{2.14b}$$

$$\mathbf{r}_3 = k^{-1}\{-\xi^3, 0, \xi^1, r^{-1}\xi^2, 0\}. \tag{2.14c}$$

With this choice the transport equations become (see appendix)

$$\begin{aligned} d\sigma_1/dt = {} & k^{-2}[-r^{-1}\xi^1\xi^2(r^{-1}v_0 - v_0') - r^{-1}\xi^2\xi^3(\alpha+\beta)]\,\sigma_1 \\ & + k^{-2}[-r^{-1}\xi^1\xi^2\beta + r^{-1}\xi^2\xi^3(r^{-1}v_0 + v_0') + (\xi^3)^2 w_0']\,\sigma_2 \\ & + k^{-2}[-\xi^1\xi^3(r^{-1}v_0 + v_0') - (r^{-1}\xi^2)^2\beta + (\xi^3)^2\alpha]\,\sigma_3, \end{aligned} \tag{2.15}$$

$$\begin{aligned} d\sigma_2/dt = {} & k^{-2}[-r^{-1}\xi^1\xi^2\alpha + (r^{-1}\xi^2)^2 w_0' - 2r^{-1}\xi^2\xi^3 v_0' - (\xi^3)^2 w_0']\,\sigma_1 \\ & + k^{-2}[\xi^1\xi^3\alpha + (r^{-1}\xi^2)^2(r^{-1}v_0 - v_0') \\ & - 2r^{-1}\xi^2\xi^3 w_0' + (\xi^3)^2(r^{-1}v_0 + v_0')]\,\sigma_3, \end{aligned} \tag{2.16}$$

$$\begin{aligned} d\sigma_3/dt = {} & k^{-2}[-r^{-1}\xi^1\xi^2 w_0' + 2r^{-1}\xi^1\xi^3 v_0 - (r^{-1}\xi^2)^2\alpha + (\xi^3)^2\beta]\,\sigma_1 \\ & + k^{-2}[\xi^1\xi^3\beta - (r^{-1}\xi^2)^2(r^{-1}v_0 - v_0') + r^{-1}\xi^2\xi^3 w_0' - 2r^{-1}(\xi^3)^2 v_0]\,\sigma_2 \\ & + k^{-2}[\xi^1\xi^3 w_0' + r^{-1}\xi^2\xi^3(\alpha+\beta)]\,\sigma_3. \end{aligned} \tag{2.17}$$

These transport equations are valid along the rays (2.12). Thus, substituting (2.12) into (2.15)–(2.17), we obtain a closed linear system of ordinary differential equations

$$d\boldsymbol{\sigma}/dt = \mathbf{A}(t)\,\boldsymbol{\sigma} \tag{2.18}$$

for the amplitude $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ of the gravity waves. This is the basic system of equations for our stability analysis. In fact, from Eckhoff (1975) it follows that the basic flow (2.2) cannot be stable unless the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) is stable for almost every possible choice of the parameters r_0, ϕ_0, z_0, ξ_0^1, ξ_0^2 and ξ_0^3 (see appendix).

3. Discussion of stability

The system (2.18) is easily seen to be autonomous if and only if

$$r_0^{-1}\xi_0^2[r_0^{-1}v_0(r_0) - v_0'(r_0)] - \xi_0^3 w_0'(r_0) = 0. \tag{3.1}$$

After a considerable amount of algebra, the eigenvalues of the matrix $\mathbf{A}$ in (2.18) are in this case found to be

$$\lambda_1 = 0, \quad \lambda_2 = ik^{-1}D, \quad \lambda_3 = -\lambda_2, \tag{3.2}$$

where $i = \sqrt{-1}$ and D is given by

$$D^2 = -(r_0^{-1}\xi_0^2)^2\alpha\beta - 2r_0^{-2}\xi_0^2\xi_0^3 v_0 w_0' - (\xi_0^3)^2[\alpha\beta - 2r_0^{-1}v_0(r_0^{-1}v_0 + v_0')]. \tag{3.3}$$

In (3.3) it is assumed that $r = r_0$ has been substituted into α, β, v_0 and w_0.

From the standard theory of stability (see Roseau 1966, p. 22) we conclude that a necessary condition for stability of the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) when (3.1) is satisfied is that $D^2 \geqslant 0$. We shall discuss this condition further below.

We now consider the cases where (3.1) is not satisfied. In these cases (2.18) is a non-autonomous system where $\mathbf{A}(t) \to 0$ as $t \to +\infty$. By introducing $\tau = \ln t$ as a new independent variable in (2.18), we obtain a system where the trivial solution has exactly the same stability properties as the trivial solution of (2.18). Asymptotically as $\tau \to +\infty$ this transformed system tends to a system with the following constant coefficient matrix:

$$\mathbf{B}_0 = \lim_{t\to\infty} t\mathbf{A}(t) = \frac{1}{e}\begin{Bmatrix} -r_0^{-1}\xi_0^2(r_0^{-1}v_0 - v_0') & -r_0^{-1}\xi_0^2\beta & -\xi_0^3(r_0^{-1}v_0 + v_0') \\ -r_0^{-1}\xi_0^2\alpha & 0 & \xi_0^3\alpha \\ -r_0^{-1}\xi_0^2 w_0' + 2r_0^{-1}\xi_0^3 v_0 & \xi_0^3\beta & \xi_0^3 w_0' \end{Bmatrix}, \tag{3.4}$$

where
$$e = r_0^{-1}\xi_0^2(r_0^{-1}v_0 - v_0') - \xi_0^3 w_0'. \tag{3.5}$$

The eigenvalues of the matrix $\mathbf{B}_0$ are found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\tfrac{1}{2} + \{\tfrac{1}{4} - (D/e)^2\}^{\frac{1}{2}}, \quad \lambda_3 = -\tfrac{1}{2} - \{\tfrac{1}{4} - (D/e)^2\}^{\frac{1}{2}}. \tag{3.6}$$

Again from the standard theory of stability, we conclude that a necessary condition for stability of the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) when (3.1) is not satisfied also is that $D^2 \geqslant 0$. Thus we may conclude that, in order that the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) shall be stable, it is always necessary that $D^2 \geqslant 0$. This implies that the basic flow (2.2) cannot be stable unless $D^2 \geqslant 0$ for every possible choice of the parameters $r_0, \phi_0, z_0, \xi_0^1, \xi_0^2$ and ξ_0^3 (see appendix and Eckhoff 1975).

Since D^2 is a quadratic form with respect to $r_0^{-1}\xi_0^2$ and ξ_0^3, it can be transformed to a diagonal form
$$D^2 = \kappa_1 x^2 + \kappa_2 y^2 \tag{3.7}$$
by an orthogonal transformation $(r_0^{-1}\xi_0^2, \xi_0^3) \to (x, y)$. The coefficients κ_1 and κ_2 are then the eigenvalues of the symmetric matrix associated with the quadratic form (3.3) and are found to be

$$\kappa_n = -\alpha\beta + r_0^{-1}v_0(r_0^{-1}v_0 + v_0') + (-1)^n\{(r_0^{-1}v_0)^2[(r_0^{-1}v_0 + v_0')^2 + w_0'^2]\}^{\frac{1}{2}} \quad (n = 1, 2). \tag{3.8}$$

In order that $D^2 \geqslant 0$ for every possible choice of the parameters r_0, ϕ_0, z_0, ξ_0^1, ξ_0^2 and ξ_0^3 it is obviously necessary and sufficient that $\kappa_1, \kappa_2 \geqslant 0$ for every possible value of r_0, i.e. for $a \leqslant r_0 \leqslant b$. Thus we can conclude that the basic flow (2.2) cannot be stable unless
$$\alpha\beta \leqslant r_0^{-1}v_0(r_0^{-1}v_0 + v_0') - \{(r_0^{-1}v_0)^2[(r_0^{-1}v_0 + v_0')^2 + w_0'^2]\}^{\frac{1}{2}} \tag{3.9}$$
holds everywhere in the fluid, i.e. for $a \leqslant r_0 \leqslant b$. Here we have that
$$\alpha\beta = -(r_0^{-1}v_0^2 + V')[\rho_0^{-1}\rho_0' - c_0^{-2}(r_0^{-1}v_0^2 + V')] = -N^2, \tag{3.10}$$
where N is the analogue of the local Brunt–Väisälä frequency (see Eckart 1960, p. 60).

From the standard theory of stability it is not difficult to show that $D^2 > 0$ is a sufficient condition to ensure stability of the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18). Furthermore, we see that $\mathbf{A}(t) \equiv 0$ in (2.18) if $\xi_0^2 = \xi_0^3 = 0$, hence $\boldsymbol{\sigma} = 0$ is stable in that case. In conclusion, we find that the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) is always stable when the strict inequality holds in (3.9).

If, on the other hand, equality holds in (3.9), we find that $D = 0$ for some values of ξ_0^2, ξ_0^3 with $(\xi_0^2)^2 + (\xi_0^3)^2 \neq 0$. In these marginal cases the eigenvalues (3.2) and (3.6) are no longer simple; therefore a more detailed analysis is needed in order to settle the stability problem for (2.18). We consider the following two cases:

$$(a) \quad r_0^{-1}v_0 - v_0' = 0, \qquad (b) \quad r^{-1}v_0 - v_0' \neq 0. \tag{3.11}$$

In case (*a*) we see that (2.18) is autonomous when $\xi_0^3 = 0$. When equality holds in (3.9), the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) will therefore be unstable for some choices of ξ_0^1, ξ_0^2 and ξ_0^3 unless

$$\mathbf{A}|_{\xi_0^3=0} = 0 \quad \text{for all} \quad \xi_0^1, \xi_0^2. \tag{3.12}$$

This is easily seen to imply that $\alpha = \beta = w_0' = 0.$ (3.13)

When (3.13) is satisfied in case (*a*), we see that (2.18) is autonomous for any choice of ξ_0^1, ξ_0^2 and ξ_0^3. Thus the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) will be unstable for some choices of ξ_0^1, ξ_0^2 and ξ_0^3 unless

$$\mathbf{A} = 0 \quad \text{for all} \quad \xi_0^1, \xi_0^2, \xi_0^3. \tag{3.14}$$

This is easily seen to imply that

$$\alpha = \beta = w_0' = r^{-1}v_0 = v_0' = 0. \tag{3.15}$$

In case (*b*) we see that (2.18) is autonomous when

$$r^{-1}\xi_0^2 = \xi_0^3 w_0'/(r^{-1}v_0 - v_0'). \tag{3.16}$$

When equality holds in (3.9), the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) will therefore be unstable for some choices of ξ_0^1, ξ_0^2 and ξ_0^3 unless

$$\mathbf{A} = 0 \quad \text{for all} \quad \xi_0^1, \xi_0^2, \xi_0^3 \quad \text{satisfying (3.16)}. \tag{3.17}$$

This is easily seen to imply that $w_0' = 0.$ (3.18)

When (3.18) is satisfied in case (*b*), we see that (2.18) is always autonomous when $r^{-1}\xi_0^2 = 0$. Thus the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) will be unstable for some choices of ξ_0^1, ξ_0^2 and ξ_0^3 unless

$$\mathbf{A}|_{\xi_0^2=0} = 0 \quad \text{for all} \quad \xi_0^1, \xi_0^3. \tag{3.19}$$

This is easily seen to imply that (3.15) must be satisfied, which in case (*b*) is not possible.

In conclusion, we have shown that, when equality holds in (3.9), the trivial solution $\boldsymbol{\sigma} = 0$ of (2.18) will be unstable for some choices of ξ_0^1, ξ_0^2 and ξ_0^3 unless (3.15) is satisfied.

In view of the above discussion and the theory in Eckhoff (1975) we have now established the following result (when $c_0 < \infty$).

THEOREM. In order that the basic flow (2.2) shall be stable it is necessary that

$$(r^{-1}v_0^2 + V')\{\rho_0^{-1}\rho_0' - c_0^{-2}(r^{-1}v_0^2 + V')\} \geqslant -r^{-1}v_0(r^{-1}v_0 + v_0') + \{(r^{-1}v_0)^2[(r^{-1}v_0 + v_0')^2 + w_0'^2]\}^{\frac{1}{2}} \tag{3.20}$$

holds everywhere in the fluid. If equality holds in (3.20) on some set of positive measure, it is further necessary for stability of the flow (2.2) that

$$v_0 = v_0' = w_0' = \rho_0' = V' = 0 \tag{3.21}$$

holds almost everywhere on this set.

4. Discussion of the results

Obviously it is not very restrictive to assume that our basic flow (2.2) does not satisfy (3.21) on any set of positive measure; we shall therefore limit our discussion to such basic flows. With the notation (3.10), the above theorem then says that the basic flow (2.2) is unstable unless

$$N^2 > -r^{-1}v_0(r^{-1}v_0 + v_0') + \{(r^{-1}v_0)^2[(r^{-1}v_0 + v_0')^2 + w_0'^2]\}^{\frac{1}{2}} \tag{4.1}$$

holds almost everywhere in the fluid. The right-hand side of (4.1) is obviously always non-negative, therefore the Brunt–Väisälä frequency N has to be a real quantity everywhere in order that the basic flow (2.2) shall be stable. More specifically, (4.1) implies that

$$N^2 > \begin{cases} -2r^{-1}v_0(r^{-1}v_0+v_0') > 0 \quad \text{when} \quad r^{-1}v_0(r^{-1}v_0+v_0') < 0, & (4.2a) \\ 0 \quad \text{when} \quad r^{-1}v_0(r^{-1}v_0+v_0') \geqslant 0. & (4.2b) \end{cases}$$

The inequality (4.1) is more restrictive than (4.2) when $w_0' \neq 0$, while (4.1) and (4.2) are equivalent when $w_0' = 0$.

As an illustration, let us consider the following family of velocity profiles:

$$v_0 = c_1 r^{\mu}, \quad w_0 = c_2, \tag{4.3}$$

where c_1, c_2 and μ are arbitrary constants ($c_1 \neq 0$). With (4.3) the criterion (4.2) [or equivalently (4.1)] becomes

$$N^2 > \begin{cases} -2(1+\mu)c_1^2 r^{2\mu-2} \quad \text{when} \quad \mu < -1, & (4.4a) \\ 0 \quad \text{when} \quad \mu \geqslant -1. & (4.4b) \end{cases}$$

In particular, when $\mu \geqslant -1$ (i.e. when the inner regions of the fluid do not rotate too fast compared with the outer regions) our criterion is completely analogous to the stability criterion for the static equilibrium of a compressible fluid in a gravitational field (see Eckart 1960, p. 60). This analogy provides an immediate physical interpretation of our criterion. In our case as well as in the static-equilibrium case, it is possible to show that the perturbations will have an exponential growth when $N^2 < 0$ while they will have a linear growth in the marginal case $N = 0$.

Under certain restrictions Gans (1975) obtained a Richardson number criterion for linear stability of the gas flows considered in this paper. Warren (1975) showed that these restrictions may be relaxed if the criterion is slightly modified. In fact, Warren (1975) showed that the basic flow (2.2) is stable if

$$N^2 > \tfrac{1}{4}\{(v_0' - r^{-1}v_0)^2 + w_0'^2\} \tag{4.5}$$

holds everywhere in the fluid. A simple modification of (4.5) reads

$$N^2 > -r^{-1}v_0(r^{-1}v_0+v_0') + \tfrac{1}{4}\{(r^{-1}v_0+v_0')^2 + w_0'^2\} + (r^{-1}v_0)^2. \tag{4.6}$$

We see that

$$[\tfrac{1}{4}\{(r^{-1}v_0+v_0')^2 + w_0'^2\} + (r^{-1}v_0)^2]^2 - (r^{-1}v_0)^2[(r^{-1}v_0+v_0')^2 + w_0'^2] = [\tfrac{1}{4}\{(r^{-1}v_0+v_0')^2 + w_0'^2\} - (r^{-1}v_0)^2]^2 \geqslant 0. \tag{4.7}$$

This shows that the necessary condition for stability (4.1), as expected, is never more restrictive than the sufficient condition (4.5). Furthermore, (4.7) shows that the necessary condition (4.1) and the sufficient condition (4.5) coincide if and only if

$$(r^{-1}v_0+v_0')^2 + w_0'^2 - 4(r^{-1}v_0)^2 = 0. \tag{4.8}$$

Thus if the basic flow (2.2) satisfies (4.8), it is stable if and only if (4.5) is satisfied. On the other hand, if the basic flow (2.2) does not satisfy (4.8), there is a gap between the sufficient condition (4.5) and the necessary condition (4.1) where the stability properties of the basic flow (2.2) still are unknown.

We note that there are several classes of velocity profiles which satisfy (4.8). This is easily verified for the following:

$$v_0 = c_1 r^n, \quad w_0 = c_2 r^n + c_3, \tag{4.9}$$

where c_1, c_2, c_3 and n are arbitrary constants satisfying

$$n^2 c_2^2 + \{(n+1)^2 - 4\} c_1^2 = 0. \tag{4.10}$$

Thus we see that if we choose c_1, c_2 and c_3 arbitrarily there are two possible values for n:

$$n_{\pm} = \frac{-c_1^2 \pm (4c_1^2 + 3c_1^2 c_2^2)^{\frac{1}{2}}}{c_1^2 + c_2^2}. \tag{4.11}$$

In particular, if we choose $c_2 = 0$, n is equal to 1 or -3 from (4.11) for any choice of c_1 and c_3.

In view of the above discussion we see that the gap between the sufficient condition (4.5) and the necessary condition (4.1) can be made arbitrarily small if the velocity profiles are restricted to be sufficiently close to rigid-body rotation, i.e. if w_0 is small compared with v_0 and v_0 is close to $c_1 r$. This explains why the asymptotic analysis done by Gans (1975) leads to a criterion giving both necessary and sufficient conditions for stability.

Howard (1973) showed that the basic flow (2.2) is stable to *axisymmetric* perturbations if

$$N^2 > \tfrac{1}{4} w_0'^2 - 2r^{-1} v_0 (r^{-1} v_0 + v_0') \tag{4.12}$$

holds everywhere in the fluid. A simple modification of (4.12) reads

$$N^2 > \tfrac{1}{4}\{(v_0' - r^{-1} v_0)^2 + w_0'^2\} - \tfrac{1}{4}(v_0' + 3r^{-1} v_0)^2. \tag{4.13}$$

This shows that (4.12), as expected, is never more restrictive than (4.5). Furthermore (4.13) shows that (4.5) and (4.12) coincide if and only if

$$v_0' + 3r^{-1} v_0 = 0, \quad \text{i.e.} \quad v_0 = c r^{-3} \quad \text{where} \quad c = \text{constant}. \tag{4.14}$$

Since Howard (1973) restricted the perturbations to be axisymmetric, a comparison of the conditions (4.1) and (4.12) is of secondary interest only. However, we see that

$$[\tfrac{1}{4} w_0'^2 - r^{-1} v_0 (r^{-1} v_0 + v_0')]^2 - (r^{-1} v_0)^2 [(r^{-1} v_0 + v_0')^2 + w_0'^2] = \tfrac{1}{4} w_0'^2 [\tfrac{1}{4} w_0'^2 - 2r^{-1} v_0 (v_0' + 3r^{-1} v_0)]. \tag{4.15}$$

Thus we obtain the following results: (4.1) is more restrictive than (4.12) if and only if

$$\tfrac{1}{4} w_0'^2 - r^{-1} v_0 (r^{-1} v_0 + v_0') < 0 \tag{4.16}$$

or

$$\tfrac{1}{4} w_0'^2 - 2r^{-1} v_0 (v_0' + 3r^{-1} v_0) < 0 \quad \text{and} \quad w_0' \neq 0. \tag{4.17}$$

Furthermore, (4.15) shows that (4.1) and (4.12) coincide if and only if

$$\tfrac{1}{4} w_0'^2 - r^{-1} v_0 (r^{-1} v_0 + v_0') \geqslant 0 \tag{4.18}$$

and either

$$w_0' = 0 \tag{4.19}$$

or

$$\tfrac{1}{4} w_0'^2 - 2r^{-1} v_0 (v_0' + 3r^{-1} v_0) = 0. \tag{4.20}$$

Appendix. The generalized progressing-wave expansion method

For symmetric hyperbolic systems an extensive literature exists (see Courant & Hilbert 1962). The generalized progressing-wave expansion method enables us to calculate approximate solutions of such systems. The method is a generalization of the WKB method, but in contrast to that method the error made by truncating the generalized progressing-wave expansion is completely understood. We shall give here a very brief discussion of a special version of the generalized progressing-wave expansion method and its application to stability problems. Further details can be found in Eckhoff (1975).

We consider a linear symmetric hyperbolic system of the form

$$\mathbf{Lu} = \mathbf{u}_t + \sum_{\nu=1}^{n} \mathbf{A}^\nu \mathbf{u}_{x_\nu} + \mathbf{Bu} = 0, \tag{A 1}$$

where $\mathbf{u} = \{u_1, \ldots, u_m\}$ are the dependent variables (i.e. the unknown functions), while $\mathbf{B}$ and $\mathbf{A}^\nu$ $(\nu = 1, \ldots, n)$ are given $m \times m$ matrices with real coefficients which may depend on the independent variables t (time) and $\mathbf{x} = \{x_1, \ldots, x_n\}$ (space). The matrices $\mathbf{A}^\nu$ $(\nu = 1, \ldots, n)$ are assumed to be symmetric.

The formal expansion

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=0}^{\infty} \frac{1}{(i\omega)^j} \mathbf{a}_j(\mathbf{x}, t) \exp\{i\omega\phi(\mathbf{x}, t)\} \tag{A 2}$$

is a generalized progressing-wave solution of (A 1) if it satisfies (A 1) to every order in *the frequency parameter* ω. The scalar function $\phi(\mathbf{x}, t)$ is called *the phase function*, the m-dimensional vector function $\mathbf{a}_0(\mathbf{x}, t)$ is *the amplitude* of the leading term, and the m-dimensional vector functions $\mathbf{a}_j(\mathbf{x}, t)$ $(j = 1, 2, \ldots)$ are called *the distortion coefficients*.

By substituting (A 2) into (A 1) and equating coefficients, we obtain the following equations:

$$\left\{\phi_t \mathbf{I} + \sum_{\nu=1}^{n} \phi_{x_\nu} \mathbf{A}^\nu\right\} \mathbf{a}_0 = 0, \tag{A 3}$$

$$\left\{\phi_t \mathbf{I} + \sum_{\nu=1}^{n} \phi_{x_\nu} \mathbf{A}^\nu\right\} \mathbf{a}_j + \mathbf{La}_{j-1} = 0 \quad (j = 1, 2, \ldots). \tag{A 4}$$

Since we assume that $\mathbf{a}_0 \neq 0$, (A 3) implies that the phase function must satisfy the characteristic equation

$$\det\left\{\phi_t \mathbf{I} + \sum_{\nu=1}^{n} \phi_{x_\nu} \mathbf{A}^\nu\right\} = 0, \tag{A 5}$$

which is a partial differential equation of order 1 and degree m. On introducing the notation

$$\lambda = -\phi_t, \quad \xi^\nu = \phi_{x_\nu} \quad (\nu = 1, \ldots, n), \quad \mathbf{E} = \sum_{\nu=1}^{n} \xi^\nu \mathbf{A}^\nu, \tag{A 6}$$

(A 5) shows that λ must be an eigenvalue of the symmetric matrix $\mathbf{E}$. If

$$\lambda = \Omega(\mathbf{x}, t, \xi^1, \ldots, \xi^n)$$

is an eigenvalue of $\mathbf{E}$, we see that (A 5) is satisfied when

$$\phi_t + \Omega(\mathbf{x}, t, \phi_{x_1}, \ldots, \phi_{x_n}) = 0. \tag{A 7}$$

The eigenvalues of the matrix $\mathbf{E}$ are therefore called the characteristic roots associated with (A 1). The Cauchy problem for (A 7) may be solved uniquely by the method of characteristics, i.e. by solving the ray equations associated with (A 7) (see Courant & Hilbert 1962, chap. 2):

$$dx_\nu/dt = \partial\Omega/\partial\xi^\nu, \quad d\xi^\nu/dt = -\partial\Omega/\partial x_\nu \quad (\nu = 1, \ldots, n). \tag{A 8}$$

To the different characteristic roots there correspond different families of phase functions which again correspond to the different classes of waves described by (A 1).

Now let Ω be a fixed eigenvalue of multiplicity μ, say, and suppose that ϕ satisfies (A 7). Equation (A 3) then shows that

$$\mathbf{a}_0 = \sum_{l=1}^{\mu} \sigma_l \mathbf{r}_l, \tag{A 9}$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_\mu$ are orthonormal eigenvectors associated with the eigenvalue Ω, and $\sigma_1, \ldots, \sigma_\mu$ are scalar functions to be determined. Upon setting $j = 1$ in (A 4), we have

$$\left\{\phi_t \mathbf{I} + \sum_{\nu=1}^{n} \phi_{x_\nu} \mathbf{A}^\nu\right\} \mathbf{a}_1 + \mathbf{L}\mathbf{a}_0 = 0. \tag{A 10}$$

This may be considered as a system of linear algebraic equations for $\mathbf{a}_1$. It has a solution if and only if

$$\mathbf{r}_l . \mathbf{L}\,\mathbf{a}_0 = 0 \quad (l = 1, \ldots, \mu). \tag{A 11}$$

Substituting (A 9) into (A 11) yields the following system of partial differential equations for the σ_l $(l = 1, \ldots, \mu)$:

$$(\sigma_l)_t + \sum_{\nu=1}^{n} \sum_{k=1}^{\mu} (\mathbf{r}_l . \mathbf{A}^\nu \mathbf{r}_k)(\sigma_k)_{x_\nu} + \sum_{k=1}^{\mu} (\mathbf{r}_l . \mathbf{L}\mathbf{r}_k)\sigma_k = 0. \tag{A 12}$$

From the definition of Ω and $\mathbf{r}_l$ $(l = 1, \ldots, \mu)$ it follows that

$$\mathbf{r}_l . \mathbf{A}^\nu \mathbf{r}_k = \begin{cases} 0 \quad \text{when} \quad l \neq k, & \text{(A 13a)} \\ \partial\Omega/\partial\xi^\nu \quad \text{when} \quad l = k. & \text{(A 13b)} \end{cases}$$

The system of equations (A 12) may therefore be interpreted as a system of ordinary differential equations

$$\frac{d}{dt}\sigma_l = -\sum_{k=1}^{\mu} (\mathbf{r}_l . \mathbf{L}\mathbf{r}_k)\sigma_k \tag{A 14}$$

along the rays determined by (A 8). Equations (A 14) are called *the transport equations* for the hyperbolic system (A 1).

When the amplitude $\mathbf{a}_0$ has been calculated from the transport equations (A 14), the distortion coefficients $\mathbf{a}_j$ $(j = 1, 2, 3, \ldots)$ may be calculated analogously from (A 4) (see Ludwig 1960, p. 479; Eckhoff 1975, p. 48). Thus all the terms in the expansion (A 2) may be calculated by solving ordinary differential equations and algebraic equations only. The expansion (A 2) is valid up to the nearest caustic, i.e. the points where, for instance, the rays cross each other, and as long as the rays do not hit the boundaries. Crossing of rays means that focusing phenomena appear in the leading term of the expansion (A 2). In Eckhoff (1975, p. 18) it is shown that caustics due to focusing of the wave never appear if and only if Ω is a linear function with respect to the variables $\xi^1, \ldots, \xi^n$, and that in these cases the transport equations (A 14) become

$$\frac{d}{dt}\sigma_l = -\sum_{k=1}^{\mu} \mathbf{r}_l . \left\{\frac{\partial \mathbf{r}_k}{\partial t} + \sum_{\nu=1}^{n} \mathbf{A}^\nu \frac{\partial \mathbf{r}_k}{\partial x_\nu} + \mathbf{B}\mathbf{r}_k\right\}\sigma_k + \sum_{k=1}^{\mu}\sum_{\nu=1}^{n} \frac{\partial\Omega}{\partial x_\nu} \mathbf{r}_l . \frac{\partial \mathbf{r}_k}{\partial \xi^\nu}\sigma_k \quad (l = 1, \ldots, \mu). \tag{A 15}$$

We see that (A 8) and (A 15) constitute a closed system of ordinary differential equations which determine the leading term in the generalized progressing-wave expansion (A 2).

From the theory in Ludwig (1960) it follows that the expansion (A 2) for sufficiently small t is an asymptotic expansion as $\omega \to \infty$. Eckhoff (1975, §§7 and 8) showed that in general the distortion coefficients $\mathbf{a}_j$ in (A 2) will *not* be uniform with respect to t, but that the leading term in (A 2) will be uniformly valid with respect to t. If the rays never hit the boundaries, this in particular means that on any finite time interval the leading term in (A 2) is an approximation of a family of exact solutions of (A 1) where the error can be made arbitrarily small by choosing ω sufficiently large. From this we may conclude that if the trivial solution of (A 15) is unstable then the trivial solution of (A 1) is necessarily unstable.

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